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Invariant Finsler metrics on homogeneous manifolds: II. Complex structures

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Abstract

In this paper, we study homogeneous complex Finsler spaces. We first prove that each homogeneous complex Finsler space can be written as a coset space of a Lie group with an invariant complex structure as well as an invariant complex Finsler metric. We then introduce the notion of Minkowski representations of Lie groups and Lie algebras to give a complete algebraic description for such spaces. Finally, we study symmetric complex Finsler spaces and obtain a complete classification of such spaces.

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1. Introduction

As pointed out by S S Chern, Finsler geometry is just Riemannian geometry without the quadratic restriction [5]. Recently, the study of Finsler geometry has been enhanced by the works of many geometers. In particular, the publications of a series of substantial books (cf [3, 6, 14]) have attracted more and more people to this field. The study of Finsler spaces has many applications in physics and biology [2]. As pointed out by Ingarden, in an anisotropic medium, the speed of light depends on its direction of travel. At each location x, visualize y as an arrow that emanates from x. Measure the time light takes to travel from x to the tip of y, and call the result $F(x, y)$. Then $\int_a^b F(x, y)$ represents the total time light takes to traverse a given path in this medium.

This paper is a continuation of our previous work [8]. In this paper, we will study invariant complex Finsler metrics on a homogeneous complex manifold. The study of invariant structures on homogeneous spaces is an important problem in geometry as well as in many branches of mathematics. The most remarkable work is due to Cartan, who established the theory of Riemannian symmetric spaces and particularly gave a complete classification of Riemannian symmetric spaces. Other classical and important works include the theory of homogeneous Riemannian manifolds, homogeneous complex manifolds, homogeneous Kähler manifolds and homogeneous symplectic manifolds. These facts justify our motivation to study invariant Finsler metrics on homogeneous manifolds. The real case was studied in [8]. The purpose of this paper is to study in some detail the complex case.

The main results of this paper can be summarized as follows. We first prove that a homogeneous complex Finsler space can be written as a coset space. Then we introduce a new definition—Minkowski representations of Lie groups and Lie algebras—to give a sufficient and necessary condition for a coset space to admit the structure of the homogeneous complex Finsler space. When the groups are complex Lie groups, the treatment is similar to the real case. But there exists homogeneous complex spaces which cannot be written as the coset space of a complex Lie group. This case is specially treated and we also get a complete algebraic description. Finally, we study symmetric complex Finsler spaces and obtain a complete list of the manifolds, which admits the structure of symmetric complex non-Riemannian Finsler spaces.

The arrangement of the paper is as follows. In section 1, we give the fundamental definitions and study homogeneous complex Finsler spaces. In section 2, we introduce the notion of Minkowski representation of Lie groups and Lie algebras; some interesting examples are also given. In section 3, we use the notion of Minkowski representations to give an algebraic description of invariant complex Finsler metrics on homogeneous manifolds. Finally, in section 4, we study symmetric complex Finsler spaces.

2. Homogeneous complex Finsler manifolds

A complex Finsler manifold (M, J, F) is a (connected) complex manifold (M, J) endowed with a complex Finsler metric F [1]. Here, by a complex Finsler metric, we mean a continuous function $F : TM \to \mathbb{R}^+$ (where $T_x(M), x \in M$ is viewed as a complex vector space) which satisfies the following conditions:

(1) F is smooth on $TM - \{0\}$;
(2) $F(u) > 0, \forall u \neq 0$;
(3) $F(\lambda u) = |\lambda| F(u)$ for any $\lambda \in \mathbb{C}^*$.

The restriction of F to any T_xM is a complex Minkowski norm, i.e., a functional on the complex vector space T_xM which is smooth on $T_xM - \{0\}$ and satisfies conditions (2) and (3). In this paper, we will only consider those complex Finsler metrics which are strongly convex as a real Finsler metric. Namely, let $x \in M$ and $v_1, v_2, \ldots, v_{2m}$ be a basis of $T_x(M)$ over the field of real numbers. For $y = y^j v_j \in T_x(M)$, write $F(y) = F(y^1, y^2, \ldots, y^{2m})$. Then the Hessian matrix

$$(g_{ij}) = \left(\tfrac{1}{2}\left[F^2_{y^i y^j}\right]\right)$$

is positive definite at any point in $T_x(M) - \{0\}$. Therefore, F is a real Finsler metric in the usual sense [3].

We first give some examples of complex Minkowski norms and complex Finsler metrics.

Example 1.1. Let (M, J, g) be a Hermitian manifold. Then it is obvious that the function F defined by

$$F(v) = \sqrt{g(v, v)}$$

is a complex Finsler metric on (M, J). In this case F is called associated with the Hermitian metric g.

Example 1.2. Let us give an example which is not associated with any Hermitian metric. Let $n \geqslant 2$. Then in the linear vector space $\mathbb{C}^n$, we define

$$F(x) = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 + \sqrt[s]{|x_1|^{2s} + |x_2|^{2s} + |x_n|^{2s}}},$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, $|\cdot|$ denotes the length of a complex number and $s \geqslant 2$ is an integer. Then it is easily seen that F is a complex Minkowski norm which is not the norm of any Hermitian form. In an obvious way, we can view $(\mathbb{C}^n, F)$ as a complex Finsler manifold. Then F is not associated with any Hermitian metric on $\mathbb{C}^n$.

Let (M, J, F) be an n-dimensional complex Finsler manifold. Then (M, F) is a $2n$-dimensional real Finsler space. In [7], we studied the group of isometries of a Finsler space. We proved that a mapping σ of M into itself is an isometry (i.e., F is a diffeomorphism such that $F(\mathrm{d}\sigma(y)) = F(y), \forall y \in TM$) of (M, F) if and only if σ is a distance-preserving mapping of M onto itself. Using this result, we proved that the group of isometries of (M, F), denoted by $I(M, F)$, is a Lie transformation group of M with respect to the compact-open topology and for any $x \in M$, the isotropy subgroup $I_x(M, F)$ at x is a compact subgroup of $I(M, F)$. In the complex case, a diffeomorphism τ of M is called holomorphic if $\mathrm{d}\tau \circ J = J \circ \mathrm{d}\tau$. The set of all holomorphic isometries of (M, J, F) forms a group, denoted by $A(M, J, F)$. It is obvious that $A(M, J, F)$ is a closed subgroup of $I(M, F)$. Therefore, by the classical result of Lie group theory (cf [10]), we have

Proposition 1.1. *The group $A(M, J, F)$ of holomorphic isometries of (M, J, F) is a Lie transformation group of M with respect to the compact-open topology. For any $x \in M$, the isotropy subgroup $A_x(M, J, F)$ at x is a compact subgroup of $A(M, J, F)$.*

Definition 1.1. *A complex Finsler manifold (M, J, F) is called homogeneous if the group of holomorphic isometries $A(M, J, F)$ acts transitively on M.*

By a classical result on homogeneous manifolds [10], we have

Proposition 1.2. *Let (M, J, F) be a homogeneous complex Finsler manifold. Then (M, J, F) can be written as a coset space G/H, where $G = A^0(M, J, F)$ is the unity component of the group $A(M, J, F)$ of holomorphic isometries and $H = A_x^0(M, J, F)$ is the isotropy subgroup of $A^0(M, J, F)$ at $x \in M$.*

As an application of the above propositions, we can prove

Theorem 1.3. *Let (M, J, F) be a homogeneous complex Finsler manifold. Then there exists a Riemannian metric g on M such that (M, J, g) is a homogeneous Hermitian manifold.*

Proof. By proposition 1.2, M can be written as the coset space G/H of a Lie group G with G-invariant complex structure J and complex Finsler metric F. Let $o = eH$ be the origin of G/H and $V = T_o(G/H)$. It is shown in [7] that the group K of linear isometries of the (real) Minkowski space (V, F) is a compact subgroup of $GL(V)$. Fix any inner product $\langle , \rangle_0$ on V, we define an inner product $\langle , \rangle$ by

$$\langle u, v\rangle = \int_K (\langle \mathrm{Ad}(k)u, \mathrm{Ad}(k)v\rangle_0 + \langle J(\mathrm{Ad}(k)u), J(\mathrm{Ad}(k)v)\rangle_0)\, \mathrm{d}k,$$

where $\mathrm{d}k$ is the standard invariant Haar measure of K. It is obvious that $\langle , \rangle$ is K-invariant, since $H \subset K$, $\langle , \rangle$ is also H-invariant. Therefore $\langle , \rangle$ can be extended to a G-invariant Riemannian metric g on G/H [11]. By the definition, it is easy to check that

$$g(JX, JY) = g(X, Y)$$

for any $X, Y \in T_x(G/H), x \in G/H$. Therefore $(G/H, J, g)$ is a homogeneous Hermitian manifold. □

By the results of this section, to study homogeneous complex Finsler manifolds we only need to consider coset spaces.

3. Minkowski representation of Lie groups and Lie algebras

To study the invariant Finsler metric on coset spaces, we have introduced several new notions such as Minkowski Lie pairs [8], Minkowski Lie algebras [8], Minkowski symmetric Lie algebras [9]. In this paper, we use a unified method to describe these notions. This will be useful to describe homogeneous complex Finsler spaces. In the following, vector spaces are assumed to be finite dimensional.

Definition 2.1. *Let G be a Lie group and (V, ρ) be a (real or complex) representation of G. If F is a (real or complex) Minkowski norm on V such that*

$$F(\rho(g)v) = F(v), \qquad \forall g \in G, \quad v \in V.$$

Then we call (V, ρ, F) a Minkowski representation of G.

To define the notion of Minkowski representations of Lie algebras, we need to recall some notations. Let (V, F) be a (real or complex) Minkowski space. Then we have two tensors, namely the fundamental form $\{g_y\}$, $y \in V - \{0\}$ and the Cartan torsion $\{C_y\}$, $y \in V - \{0\}$. They are defined by

$$g_y(u, v) = \frac{1}{2}\left[\frac{\partial^2 F^2(y + su + tv)}{\partial s \partial t}\right]\Bigg|_{s=t=0}, \quad y \neq 0, u, v \in V,$$

$$C_y(u, v, w) = \frac{1}{4}\left[\frac{\partial^3 F^2(y + su + tv + rw)}{\partial s \partial t \partial r}\right]\Bigg|_{s=t=r=0}, \quad y \neq 0, u, v, w \in V.$$

Definition 2.2. *Let $\mathfrak{g}$ be a (real or complex) Lie algebra. Then a Minkowski representation of $\mathfrak{g}$ is a representation (V, ϕ) of $\mathfrak{g}$ with a (real or complex) Minkowski norm F on the (real or complex) vector space V such that*

$$g_y(\phi(x)u, v) + g_y(u, \phi(x)v) + 2C_y(\phi(x)y, u, v) = 0,$$

for any $x \in \mathfrak{g}$, $y(\neq 0), u, v \in V$. We usually denote the Minkowski representation by (V, ϕ, F).

Since a complex vector space can be viewed as a real vector space, we can consider the representation of a real Lie algebra $\mathfrak{g}$ on a complex vector space V, where V is viewed as a real vector space and is denoted by $V_{\mathbb{R}}$. Therefore, we can define the notion of complex Minkowski representations of a real Lie algebra. We first give some examples of Minkowski representations.

Example 2.1. Let G be a Lie group, and H be a closed connected subgroup of G. Lie $HG = \mathfrak{g}$, Lie $H = \mathfrak{h}$. Suppose F is a Minkowski norm on the quotient space $\mathfrak{g}/\mathfrak{h}$ such that $(\mathfrak{g}, \mathfrak{h}, F)$ is a Minkowski Lie pair [8]; that is,

$$g_y(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)(u), v) + g_y(u, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)v) + C_y(\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)y, u, v) = 0,$$

for any $y(\neq 0), u, v \in \mathfrak{g}/\mathfrak{h}$, $x \in \mathfrak{h}$, where $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$ is the representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ induced by the adjoint representation. Then it is obvious that $(\mathfrak{g}/\mathfrak{h}, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}, F)$ is a Minkowski representation of $\mathfrak{h}$. It is proved in [8] that in this case $(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}, F)$ is a Minkowski representation of H.

Example 2.2. Let $(\mathfrak{g}, F)$ be a Minkowski Lie algebra [8]; that is, $\mathfrak{g}$ is a real Lie algebra, F is a Minkowski norm on $\mathfrak{g}$ and the following condition is satisfied:

$$g_y([x,u],v) + g_y(u,[x,v]) + 2C_y([x,y],u,v),$$

where $y(\neq 0), x, u, v \in \mathfrak{g}$. Then it is easily seen that $(\mathfrak{g}, \mathrm{ad}, F)$ is a Minkowski representation of $\mathfrak{g}$, where ad is the adjoint representation of $\mathfrak{g}$.

Example 2.3. Let $(\mathfrak{g}, \sigma, F)$ be a Minkowski symmetric Lie algebra [9]; that is, $\mathfrak{g}$ is a real Lie algebra, σ is an involutive automorphism of $\mathfrak{g}$ with canonical decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, F is a Minkowski norm on $\mathfrak{p}$ and the following condition is satisfied:

$$g_y([x,u],v) + g_y(u,[x,v]) + 2C_y([x,y],u,v), \qquad y(\neq 0), u, v \in \mathfrak{p}, x \in \mathfrak{k}.$$

Then it is easily seen that $(\mathfrak{p}, \mathrm{ad}, F)$ is a representation of $\mathfrak{k}$, where ad is the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ (cf [10]).

Example 2.4. Let us give an explicit example of the Minkowski representation of a Lie group as well as its Lie algebra. Consider the classical simple group $G = SU(n)$ with $n \geqslant 3$. On the Lie algebra $\mathfrak{su}(n)$, for each positive real number λ, we define

$$F_\lambda(A) = \sqrt{\sqrt{\sum_{i=1}^{n} |\mu_i(A)|^4} + \lambda \sum_{i=1}^{n} |\mu_i(A)|^2},$$

where $\mu_i(A), i = 1, 2, \ldots, n$ are all the eigenvalues of A and $|\cdot|$ is the length function, $A \in \mathfrak{su}(n)$. It is easy to check that $(\mathfrak{su}(n), \mathrm{Ad}, F_\lambda)$ are a series of Minkowski representations of G which are not isomorphic to each other. By theorem 2.1, $(\mathfrak{su}(n), \mathrm{ad}, F_\lambda)$ are Minkowski representations of $\mathfrak{su}(n)$.

The relation between Minkowski representations of Lie groups and Lie algebras is as follows:

Theorem 2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If (V, ρ, F) is a (real) Minkowski representation of G, then $(V, \mathrm{d}\rho, F)$ is a Minkowski representation of $\mathfrak{g}$. On the other hand, if (V, ϕ, F) is a Minkowski representation of $\mathfrak{g}$ and G is connected, then there exists a Minkowski representation (V, ρ, F) with $\phi = \mathrm{d}\rho$.*

Proof. The proof is similar to the special case of example 2.1, which can be found in our previous paper [8]. □

We also have the following:

Theorem 2.2. *Let G be a Lie group and (V, ρ, F) be a real (complex) Minkowski representation of G. Then there exists an inner product (Hermitian inner product) $\langle , \rangle$ on V such that $(V, \rho, \langle , \rangle)$ is an orthogonal (unitary) representation of G.*

Proof. Similar to theorem 1.3. Just note that $\rho(G)$ is contained in the group K of linear isometries of (V, F), which is a compact subgroup of $GL(V)$. □

Remark. The study of Minkowski representations should be an interesting problem in representation theory and it deserves research. According to some authors, the spacetime structure is not only in a state described by Riemannian geometry, but also in a state described by Finsler geometry [4]. Thus we should not restrict us to the quadratic case. As an interesting problem, we hope to find out a sufficient and necessary condition that in a unitary representation $(V, \rho, \langle , \rangle)$ there exists a non-quadratic Minkowski norm F such that (V, ρ, F) is a Minkowski representation. The notion of Minkowski representation can be generalized to the infinite-dimensional case.

4. Complex structures

In this section, we will use the notion of Minkowski representations of Lie groups and Lie algebras to study homogeneous complex Finsler spaces. By proposition 1.2, we only need to study the invariant structures on a coset space G/H, where G is a connected Lie group and H is a closed subgroup of G.

We first consider the special case when G is a complex Lie group; that is, G is a (abstract) group as well as a complex manifold and the mapping $(x, y) \mapsto xy$ respectively $x \mapsto x^{-1}$ is a holomorphic mapping from $G \times G$ respectively G to G. As a manifold, G can be viewed as a real smooth manifold, denoted by $G_{\mathbb{R}}$. Then $G_{\mathbb{R}}$ with the original group operation is a real Lie group. The Lie algebra $\mathfrak{g}_{\mathbb{R}}$ has a natural complex structure I. Therefore $\mathfrak{g}_{\mathbb{R}}$ can be viewed as a complex Lie algebra, which we denote by $\mathfrak{g}$ and call the complex Lie algebra of the complex Lie group G.

Let G be a complex Lie group and H be a closed complex subgroup of G. Then it is easily seen that the coset space G/H has a natural complex structure I such that $(G/H, I)$ is a homogeneous complex manifold. Identifying the tangent space $T_o(G/H)$ of G/H at the origin $o = eH$ with the quotient space $\mathfrak{g}/\mathfrak{h}$, we have

Theorem 3.1. *Let G be a complex Lie group and H be a closed complex subgroup of G. If F is a complex Finsler metric on G/H such that $(G/H, I, F)$ is a homogeneous complex Finsler manifold, then $(\mathfrak{g}/\mathfrak{h}, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}, F_o)$ is a Minkowski representation of $\mathfrak{h}$. On the other hand, if F_* is a complex Minkowski norm on $\mathfrak{g}/\mathfrak{h}$ such that $(\mathfrak{g}/\mathfrak{h}, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}, F_*)$ is a Minkowski representation of $\mathfrak{h}$ and the subgroup H is connected, then there exists a complex Finsler metric F on G/H such that $F_* = F_o$ and $(G/H, I, F)$ is a homogeneous complex Finsler manifold.*

Proof. The proof is similar to the real case, see [8]. □

Theorem 3.1 gives a complete description of the structure of invariant complex homogeneous Finsler metrics on the coset spaces of complex Lie groups. However, not every complex homogeneous manifold can be written as a coset space of a complex Lie group. Let us give an example.

Example 3.1. Consider a bounded domain D in $\mathbb{C}^n$. The group of holomorphic diffeomorphisms of D onto itself, denoted by $H(D)$, is a real Lie group [10]. If the action of $H(D)$ on D is transitive, then D is a homogeneous complex manifold, called a homogeneous bounded domain. However, in this case, there cannot be any complex structure on $H(D)$ which makes $H(D)$ a complex Lie group. In fact, if $H(D)$ is a complex Lie group, then the orbit of any one-parameter subgroup of $H(D)$ is bounded. According to Liouville's theorem, any bounded holomorphic function on $\mathbb{C}$ is a constant. This means that the orbit of any one-parameter subgroup of $H(D)$ consists of one single point, contracting the fact that $H(D)$ acts transitively on D.

Therefore, to obtain a complete algebraic description of all the complex homogeneous Finsler spaces, we have to consider every coset space G/H, where G is a *real* Lie group, and H is a closed subgroup of G. In the following, we will find a sufficient and necessary condition for such a coset space to have a complex structure and an invariant complex Finsler metric simultaneously.

In the following, we will encounter several representations of a Lie algebra $\mathfrak{h}$ which are all induced by the adjoint representation. To keep the notation simple, we sometimes denote such representations simply by ad. Moreover, for a linear transformation T on a real vector space

V, we use T^C to denote the induced complex linear transformation on the complexification V^C.

Theorem 3.2. *Let G be a (real) Lie group, H be a closed subgroup of G, Lie $G = \mathfrak{g}$, Lie $H = \mathfrak{h}$. If I is a complex structure on G/H and F is a Finsler metric on G/H such that $(G/H, I, F)$ is a homogeneous complex Finsler space, then there exists a complex subalgebra $\mathfrak{a}$ of the complex Lie algebra $\mathfrak{g}^C$ (complexification of $\mathfrak{g}$) satisfying $a \cap \bar{a} = \mathfrak{h}^C$, $\mathfrak{a} + \bar{\mathfrak{a}} = \mathfrak{g}^C$ and a complex Finsler metric F_1 on $\mathfrak{a}/\mathfrak{h}^C$ such that $(\mathfrak{a}/\mathfrak{h}^C, \mathrm{ad}, F_1)$ is a Minkowski representation of $\mathfrak{h}$. On the other hand, if there exists a complex subalgebra $\mathfrak{a}$ of $\mathfrak{g}^C$ satisfying $\mathfrak{a} \cap \bar{\mathfrak{a}} = \mathfrak{h}^C$, $\mathfrak{a} + \bar{\mathfrak{a}} = \mathfrak{g}^C$ and a complex Minkowski norm F_1 on $\mathfrak{a}/\mathfrak{h}^C$ such that $(\mathfrak{a}/\mathfrak{h}^C, \mathrm{ad}, F_1)$ is a Minkowski representation of $\mathfrak{h}$ and if, moreover, H is connected, then there exists a complex structure I on G/H and a complex Finsler metric F on G/H such that $(G/H, I, F)$ is a homogeneous complex Finsler space.*

Proof. First suppose that I is a G-invariant complex structure on G/H and F is a Finsler metric on G/H such that $(G/H, I, F)$ is a homogeneous complex space. Then I corresponds to a Koszul operator J [12, 13] which is a linear transformation of $\mathfrak{g}$ such that there exists a subspace $\mathfrak{m}$ satisfying

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum of subspaces)}$$

and $J(\mathfrak{h}) = 0$, $J(\mathfrak{m}) \subset \mathfrak{m}$, $J^2|_{\mathfrak{m}} = -id$. Moreover, we also have

$$\pi(JX) = I_o(\pi(X)), \quad X \in \mathfrak{g},$$
$$\mathrm{Ad}(h)J \equiv J\,\mathrm{Ad}(h), \quad (\mathrm{mod}\ \mathfrak{h}), \quad \forall h \in H,$$

where o is the origin of G/H and π is the natural projection of $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{h}$. Now we extend J to a complex linear transformation J^C of $\mathfrak{g}^C = \mathfrak{h}^C + \mathfrak{m}^C$. Define

$$\mathfrak{n}^{\pm} = \{X \in \mathfrak{g}^C | J^C(X) = \pm\sqrt{-1}X\}.$$

It is easily seen that $\mathfrak{n}^{\pm} \subset \mathfrak{m}^C$. Let $\mathfrak{a} = \mathfrak{h}^C + \mathfrak{n}^+$. Then it is easily seen that $\bar{\mathfrak{a}} = \mathfrak{h}^C + \mathfrak{n}^-$. Therefore we have $\mathfrak{a} \cap \bar{\mathfrak{a}} = \mathfrak{h}^C$, $\mathfrak{g} = \mathfrak{a} + \bar{\mathfrak{a}}$. Furthermore, by the definition of $\mathfrak{a}$, for any $X \in \mathfrak{h}^C$, $Y \in \mathfrak{a}$, we have

$$I_o^C\pi^C([X, Y]) = I_o^C\pi^C(\mathrm{ad}(X)Y)$$
$$= I_o^C\mathrm{ad}(X)\pi^C(Y) = \mathrm{ad}(X)I_o^C(\pi^C(Y)).$$

Since $\pi^C(Y) \in \mathfrak{n}^+$, we have $I_o^C(\pi^C(Y)) = \sqrt{-1}\pi^C(Y)$. Thus

$$I_o^C(\pi^C([X, Y])) = \sqrt{-1}\,\mathrm{ad}(X)\pi^C(Y) = \sqrt{-1}\pi^C([X, Y]).$$

Therefore, we have $\pi^C([X, Y]) \in \mathfrak{n}^+$. Hence $[X, Y] \in \mathfrak{a}$; that is,

$$[\mathfrak{h}^C, \mathfrak{a}] \subset \mathfrak{a}.$$

Now for any $Y_1, Y_2 \in \mathfrak{a}$, we have

$$J^C[Y_1, Y_2] = \sqrt{-1}[Y_1, Y_2] \quad (\mathrm{mod}\ \mathfrak{h}^C).$$

Combining this fact with $[\mathfrak{h}^C, \mathfrak{a}] \subset \mathfrak{a}$, we see that $\mathfrak{a}$ is a complex subalgebra of $\mathfrak{g}^C$.

Now the Finsler metric F on G/H defines a Minkowski norm on the vector space $T_o(G/H)$. Identifying $\mathfrak{g}/\mathfrak{h}$ with $T_o(G/H)$, we obtain a Minkowski norm F^* on $\mathfrak{g}/\mathfrak{h}$ which satisfies

$$F^*(ax + bI_o(x)) = \sqrt{a^2 + b^2}F(x), \quad x \in \mathfrak{g}/\mathfrak{h}, a, b \in \mathbb{R},$$
$$F^*(\mathrm{Ad}(h)x) = F(x), \quad h \in H, x \in \mathfrak{g}/\mathfrak{h}.$$

Now we extend F^* to a Minkowski norm on $(\mathfrak{g}/\mathfrak{h})^C = \mathfrak{g}^C/\mathfrak{h}^C$ (denoted by F_1^*) by

$$F_1^*(x + \sqrt{-1}y) = \sqrt{(F^*(x))^2 + (F^*(y))^2}, \quad x, y \in \mathfrak{g}/\mathfrak{h}.$$

It is obvious that

$$F_1^*\big(\big(aid + bI_o^C\big)x\big) = \sqrt{a^2+b^2}F_1^*(x), \qquad x \in \mathfrak{g}^C/\mathfrak{h}^C, a, b \in \mathbb{R}.$$

Let F_1 be the restriction of F_1^* to $\mathfrak{a}/\mathfrak{h}^C$. Then by the definition, we see that $F_1(\mathrm{Ad}(h)(x)) = F_1(x), \forall x \in \mathfrak{a}/\mathfrak{h}^C, h \in H$. On the other hand, for any $a, b \in \mathbb{R}, x \in \mathfrak{a}$, we have

$$\begin{aligned} F_1((a + b\sqrt{-1})\pi^C(x)) &= F_1(\pi^C((a + bJ^C)x)) \\ &= F_1\big(\big(a + bI_o^C\big)\pi^C(x)\big) = \sqrt{a^2+b^2}F_1(\pi^C(x)). \end{aligned}$$

Therefore F_1 is a complex Minkowski norm on $\mathfrak{a}/\mathfrak{h}^C$. Since F_1 is invariant under H, $(\mathfrak{a}/\mathfrak{h}^C, \mathrm{Ad}, F_1)$ is a Minkowski representation of H. By theorem 2.1, $(\mathfrak{a}/\mathfrak{h}^C, \mathrm{ad}, F_1)$ is a Minkowski representation of $\mathfrak{h}$.

Conversely, if there exists a complex subalgebra $\mathfrak{a}$ satisfying $\mathfrak{g}^C = \mathfrak{a}+\bar{\mathfrak{a}}$, $\mathfrak{a}\cap\bar{\mathfrak{a}} = \mathfrak{h}^C$ and a complex Minkowski norm F_1 on $\mathfrak{a}/\mathfrak{h}^C$ such that $(\mathfrak{a}/\mathfrak{h}^C, \mathrm{ad}, F_1)$ is a Minkowski representation of $\mathfrak{h}$, then we have a direct decomposition

$$(\mathfrak{g}/\mathfrak{h})^C = \pi^C(\mathfrak{a}) + \pi^C(\bar{\mathfrak{a}}).$$

Since $\pi^C(\bar{\mathfrak{a}}) = \overline{\pi^C(\mathfrak{a})}$, we can define a (real) linear transformation I_o on the real vector space $\mathfrak{g}/\mathfrak{h}$ such that $I_o^C\big|_{\pi^C(\mathfrak{a})} = \sqrt{-1}id$, $I_o^C\big|_{\pi^C(\bar{\mathfrak{a}})} = -\sqrt{-1}id$. Then it is known that I_o can be extended to an almost complex structure I on G/H and, moreover, I is integrable if and only if $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}^C$ [12]. Thus we have defined a complex structure on G/H. Now we extend the Minkowski norm F_1 (denoted by F_1^*) to $\mathfrak{g}^C/\mathfrak{h}^C = (\mathfrak{g}/\mathfrak{h})^C$ by

$$F_1^*((x + \mathfrak{h}^C) + (\bar{y} + \mathfrak{h}^C)) = \sqrt{(F_1(x + \mathfrak{h}^C))^2 + (F_1(y + \mathfrak{h}^C))^2}, \quad x, y \in \mathfrak{a}.$$

It is easily seen that F_1^* is a complex Minkowski norm on $\mathfrak{g}^C/\mathfrak{h}^C$. Let F_o be the restriction of F_1^* to $\mathfrak{g}/\mathfrak{h}$, and for $x \in \mathfrak{g}$ let

$$\pi^C(x) = (x_1 + \mathfrak{h}^C) + (\bar{x}_2 + \mathfrak{h}^C),$$

where $x_1, x_2 \in \mathfrak{a}$. Then for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned} F((a + b\sqrt{-1})\pi(x)) &= F_o((a + bI_o)\pi(x)) \\ &= F_1^*\big(\big(a + bI_o^C\big)\pi^C(x)\big) = F_1^*\big(\big(a + bI_o^C\big)((x_1 + \mathfrak{h}^C) + (\bar{x}_2 + \mathfrak{h}^C))\big) \\ &= F_1^*((a + b\sqrt{-1})(x_1 + \mathfrak{h}^C) + (a - b\sqrt{-1})(\bar{x}_2 + \mathfrak{h}^C)) \\ &= \sqrt{(F_1((a + b\sqrt{-1})(x_1 + \mathfrak{h}^C)))^2 + (F_1((a - b\sqrt{-1})(\bar{x}_2 + \mathfrak{h}^C))^2} \\ &= \sqrt{(a^2+b^2)(F_1(x_1 + \mathfrak{h}^C))^2 + (a^2+b^2)(F_1(\bar{x}_2 + \mathfrak{h}^C))^2} \\ &= \sqrt{a^2+b^2}\sqrt{(F_1(x_1 + \mathfrak{h}^C))^2 + (F_1(\bar{x} + \mathfrak{h}^C))^2} \\ &= \sqrt{a^2+b^2}F_1(\pi^C(x)) = \sqrt{a^2+b^2}F_o(\pi(x)). \end{aligned}$$

Therefore, F_o is a complex Minkowski norm on $\mathfrak{g}/\mathfrak{h}$ (with respect to the complex structure I_o). On the other hand, since $(\mathfrak{a}/\mathfrak{h}^C, \mathrm{ad}, F_1)$ is a Minkowski representation of $\mathfrak{h}$, we easily see that $(\mathfrak{g}/\mathfrak{h}, \mathrm{ad}, F_o)$ is a Minkowski representation of $\mathfrak{h}$. Since H is connected, $(\mathfrak{g}/\mathfrak{h}, \mathrm{Ad}, F_o)$ is a Minkowski representation of the group H. Thus we can define a complex Finsler metric F on G/H which is invariant under the action of G [8]. Consequently $(G/H, I, F)$ is a homogeneous complex space. □

Remark. In some special cases, we *a priori* have an invariant complex structure on G/H. In this case, to make G/H a homogeneous complex Finsler space, we only need to find an invariant complex Finsler metric on G/H.

5. Symmetric spaces

In this section, we study complex symmetric Finsler spaces. Let (M, I, F) be a complex Finsler space. Then (M, I, F) is called (globally) symmetric if for any point $x \in M$ there exists an involutive holomorphic isometry σ_x of M such that x is an isolated fixed point of σ_x. It is easily seen that in this case, the group $A(M, I, F)$ acts transitively on M. Hence (M, I, F) is a homogeneous complex Finsler space. Let $G = A_0(M, I, F)$ and H be the isotropy subgroup of G at some fixed point x in M. Then $M = G/H$. Define an automorphism σ of G by $\sigma(g) = \sigma_x \cdot g \cdot \sigma, g \in G$. Then σ is an involutive automorphism of G and $K_\sigma^0 \subset H \subset K_\sigma$, where K_σ is the subgroup of G consisting of fixed points of σ and K_σ^0 is the identity component of K_σ. This means that (G, H) is a symmetric pair. Since H is compact, (G, H) is a Riemannian symmetric pair. By the standard results on Hermitian symmetric spaces (see [10], chapter VIII), we have

Theorem 4.1. *Let $(G/H, I, F)$ be a globally symmetric complex Finsler space. Then there exists a G-invariant Riemannian metric g on G/H such that $(G/H, I, g)$ is a Hermitian symmetric space.*

Since the Hermitian symmetric spaces were completely classified by É Cartan [10], theorem 4.1 reduces the problem of classification of symmetric complex Finsler spaces to the problem of determining which Hermitian symmetric space admits an invariant non-Riemannian complex Finsler metric (see the remark at the end section 3). Now we can prove

Theorem 4.2. *Let $(G_1/H_1, I_1, g_1)$ and $(G_2/H_2, I_2, g_2)$ be two Hermitian symmetric spaces. Then on the coset space $G_1/H_1 \times G_2/H_2$ there exist infinitely many non-Riemannian Finsler metrics F which are invariant under $G_1 \times G_2$ and make $(G_1/H_1 \times G_2/H_2, I_1 \times I_2, F)$ a symmetric complex Finsler space.*

Proof. Let o_1, o_2 be the origin of $G_1/H_1, G_2/H_2$ respectively and $o = (O_1, O_2)$. For any tangent vector $y = (y_1, y_2) \in T_o(G_1/H_1 \times G_2/H_2)$, $y_i \in T_{o_i}(G_i/H_i)$, $i = 1, 2$, and any integer $s \geqslant 2$, define

$$F_o(y) = \sqrt{g_1(y_1, y_1) + g_2(y_2, y_2) + \sqrt[s]{g_1(y_1, y_1)^s + g_2(y_2, y_2)^s}}.$$

Then F_o is a non-Euclidean Minkowski norm on $T_o(G_1/H_1 \times G_2/H_2)$ which is invariant under $H_1 \times H_2$. Hence it can be extended to a Finsler metric F on $G_1/H_1 \times G_2/H_2$ [8]. It is obvious that F is non-Riemannian. Since g_1, g_2 are Hermitian metrics, for any $a, b \in \mathbb{R}$, we easily see that

$$F_o((a + bI)y) = F_o((a + bI_1)y_1, (a + bI_2)y_2) = \sqrt{a^2 + b^2} F_o(y).$$

Therefore F is a complex Minkowski norm. □

The irreducible cases can also be completely settled. First we prove

Theorem 4.3. *Let G/H be an (compact or noncompact) irreducible Hermitian symmetric space. Then there exists an invariant complex structure I on G/H such that $(G/H, I, F)$ is a symmetric complex Finsler space for any G-invariant Finsler metric on G/H.*

Proof. The only thing we need to check is that any invariant Finsler metric F on G/H must be a complex Finsler metric with respect to the complex structure. We only treat the compact case. The noncompact case can be proved using the duality. According to [10] (p 381), if

Table 1. Irreducible symmetric complex non-Riemannian Finsler spaces.

	Noncompact ones	Compact ones	Rank	Dimension
AIII	$SU(p,q)/S(U_p\times U_q)$	$SU(p+q)/S(U_p\times U_q)$	$\min(p,q)$	$2pq$
BDI	$SO_o(p,2)/SO(p)\times SO(2)$	$SO(p+2)/SO(p)\times SO(2)$	2	$2p$
DIII	$SO^*(2n)/U(n)(n\geqslant 4)$	$SO(2n)/U(n)(n\geqslant 4)$	$[\frac{1}{2}n]$	$n(n-1)$
CI	$Sp(n,\mathbb{R})/U(n)(n\geqslant 2)$	$Sp(n)/U(n)(n\geqslant 2)$	n	$n(n+1)$
EIII	$(\mathfrak{e}_{6(-14)},\mathfrak{so}(10)+\mathbb{R})$	$(\mathfrak{e}_{6(-78)},\mathfrak{so}(10)+\mathbb{R})$	2	32
EVII	$(\mathfrak{e}_{7(-25)},\mathfrak{e}_6+\mathbb{R})$	$(\mathfrak{e}_{7(-133)},\mathfrak{e}_6+\mathbb{R})$	3	54

Note: $p,q\geqslant 2$.

G/H is a compact irreducible Hermitian symmetric space, then G/H can also be written as U/K where U is a connected compact simple Lie group with centre $\{e\}$ and K is a maximal connected proper subgroup of U with a nondiscrete centre. Moreover, the centre of K, Z_K is isomorphic to the circle group S^1. Therefore there exists an element of order 4, say j, in K. Let I denote the induced linear transformation of $\mathrm{Ad}(j)$ to $T_o(U/K)$. Then $I^2=-id$ and it is true that I induces an invariant complex structure on U/K. Now suppose that F is a U-invariant Finsler metric on U/K. Let $a,b\in\mathbb{R}, a^2+b^2\neq 0$. Then we assert that the endomorphism $\frac{a}{\sqrt{a^2+b^2}}id+\frac{b}{\sqrt{a^2+b^2}}I$ lies in the image of Z_K under the inducing mapping (to $T_o(U/K)$). In fact, since $Z_K\simeq S^1$ and j is of order 4, we easily see that there exists $x\in\mathfrak{k}$ such that $Z_K=\{\exp tX|t\in\mathbb{R}\}$ and $\exp X=j$. Since $\mathrm{Ad}(j)$ induces the complex structure I on $T_o(U/K)$, we easily see that $\mathrm{Ad}(\exp(tX))$ induces the endomorphism $T(t)$ of $T_o(U/K)$, where

$$T(t)=e^{tI}=(\cos t)id+(\sin t)I,\qquad t\in\mathbb{R}.$$

This proves our assertion. Now

$$\begin{aligned}F((aid+bI)y)&=F\left(\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}id+\frac{b}{\sqrt{a^2+b^2}}I\right)y\right)\\&=\sqrt{a^2+b^2}F\left(\left(\frac{a}{\sqrt{a^2+b^2}}id+\frac{b}{\sqrt{a^2+b^2}}I\right)y\right)\\&=\sqrt{a^2+b^2}F(T(\theta)y)\\&=\sqrt{a^2+b^2}F(y),\end{aligned}$$

where $\theta=\arccos\frac{a}{\sqrt{a^2+b^2}}$ and we have used the above assertion and the fact that F is invariant under $\mathrm{Ad}(K)$. Thus we have proved that F is a complex Finsler metric on U/K. This proves the theorem. □

Szabó [15] proved that on each irreducible globally symmetric Riemannian manifold G/H there exist infinitely many invariant Finsler metrics on G/H if rank $G/H\geqslant 2$ and there does not exist any non-Riemannian invariant Finsler metric on it if rank $G/H=1$. On the other hand, the irreducible Hermitian symmetric spaces were completely classified by É Cartan (cf [10], p 518). Combining these results with theorem 4.3 we get a classification of irreducible symmetric complex non-Riemannian Finsler spaces (table 1).

Theorem 4.4. *Let G/H be an irreducible Riemannian symmetric space. If G/H admits an invariant complex structure I and a non-Riemannian Finsler metric F such that $(G/H,I,F)$ is a symmetric complex Finsler space, then G/H must be one of the manifolds in table 1. Furthermore, on each manifold in table 1 there exist infinitely many invariant complex non-Riemannian Finsler metrics.*

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